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# DEVELOPMENT OF A TOOLBOX SUPPORTING FUZZY CALCULATIONS

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**ABSTRACT:**
Alpha-cut based calculations are widely used in fuzzy arithmetic and fuzzy rule interpolation based reasoning. One of the key issues of the successful development of these applications is the availability of a toolbox that makes possible the quick and efficient calculation of the α-cuts' endpoints. In this paper, after reviewing some basic theoretical concepts, we present the methods of the α-cut calculation in case of the most used membership function types. The presented methods were also implemented in C# in form of a dynamic link library, which is easy useable in every .NET or traditional Windows or Linux targeting software applications.
**KEYWORDS:**
α-cut calculation, fuzzy set, toolbox

## 1. INTRODUCTION

Fuzzy sets can be seen as an extension of the traditional set concept. In case of a crisp (traditional) set ($A$) every $x$ element of a universe of discourse $X$ can be evaluated only two-way, either as part of the set ($x \in A$) or as not belonging to the set ($x \notin A$). Contrary to this all-or-nothing approach the fuzzy concept [11] makes possible a more colorful interpretation of boundaries. It allows the expression of the membership's measure not only by 0 and 1 but also by any value of the unit interval.

The fuzzy approach has been successfully applied in several areas of the science and the everyday life. Thus has been emerged the fuzzy arithmetic (e.g. [2],[3]) and one can find many practical applications in the field of control (e.g. [9],[4]) or fuzzy modeling of processes and systems (e.g. [6]). A huge amount of these applications does the computations α-cut wise based on Zadeh's extension principle [3]. One of the key issues of the successful practical application is the availability of a toolbox that supports the auxiliary calculations, i.e. the quick and efficient determination of the α-cuts.

In this paper, after reviewing some important theoretical concepts, we present the α-cuts' calculation methods for the most often used membership function types. The methods being presented were also implemented in C# in form of a dynamic link library, which is easy useable in every .NET or traditional Windows or Linux targeting software applications.

## 2. FUZZY SETS AND RELATED CONCEPTS

In this section we review briefly some concepts and definitions that are strongly related to the α-cut calculation and its applications.

**Universe of discourse**. Notation: $X$ or $U$.

The universe of discourse is a crisp (traditional) set, also called domain, from which the members of a fuzzy set are taken. For example the set of the real numbers ($\Re$) can be an universe of discourse.

**Fuzzy set**. Notation: capital roman letter, e.g. *A*.

The fuzzy set can be seen as an extension of the traditional set concept. While in case of the crisp sets each member of the universe of discourse can be tagged squarely as member of the set or outsider, in case of fuzzy sets one can assign a membership level as well.

**Membership function**. Notation: $\mu_A$.

The function $\mu_A : X \to [0,1]$ expresses in which measure the members of the universe *X* belong to the fuzzy set *A*.

**Normal fuzzy set**.

The fuzzy set *A* is considered normal if $\exists x \in X$, for which $\mu_A(x) = 1$.

**α-cut**. Notation: $[A]_\alpha$.

The $\alpha$-cut of a fuzzy set (*A*) is a crisp set that is defined by

$$[A]_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha; \alpha \in (0,1]\} = [\underline{a}_\alpha, \overline{a}_\alpha], \tag{1}$$

where $\underline{a}_\alpha = \inf\{ [A]_\alpha \}$ and $\overline{a}_\alpha = \sup\{ [A]_\alpha \}$ are the lower respective upper endpoints of the $\alpha$-cut.

**Convex fuzzy set**.

A fuzzy set (*A*) is convex when all of its α-cuts are convex sets

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)] \;\; \forall\, x_1, x_2 \in \Re^2,\; \lambda \in [0,1]. \tag{2}$$

**Support**. Notation: supp(*A*) or $[A]_{0+}$.

The support of a fuzzy set is defined by

$$\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}. \tag{3}$$

**Core**. Notation: core(*A*) .

The core of a fuzzy set is crisp set that contains those elements of *X* for which the membership function takes its maximum value

$$\mathrm{core}(A) = \{x \in X \mid \mu_A(x) = \max(\mu_A(x))\}. \tag{4}$$

**Compact fuzzy set.**

A fuzzy set (*A*) is compact when its support is bounded, i.e. $\exists\, x_1, x_2 \in X,\;\; \mathrm{supp}(A) \subset [x_1, x_2]$.

**Fuzzy number.**

A fuzzy set *A* ( $\mu_A : \Re \to [0,1]$) is a fuzzy number when fulfills the following requirements [2].

- The set is convex and normal.
- The membership function is at least piece-wise continuous.
- The set is compact on $\Re$ .

**Reference point**. Notation: $RP(A)$

The reference point of a fuzzy set (*A*) is that element of *X*, which is in one or more aspects characteristic to the position of *A*. The reference point is used by several fuzzy methods (e.g. fuzzy rule interpolation based inference techniques) for the characterization of a set's position. Although there are several options for its selection, usually the center point of the set's core is applied for this task [1].

**Left/Right flank.**

The point {*RP*(*A*), $\mu_A$(*RP*(*A*))} divides the membership function (set shape) into two parts called left and right flanks of the set.

## 3. ALPHA-CUT COMPUTATION

Several fuzzy methods use the set's left and right flanks separately for the calculations; furthermore in several cases one needs different α-cuts in case of the two flanks. In order to satisfy this need our toolbox calculates and handles separately the lower and upper endpoints of the cuts. The calculation methods were developed for the most frequently applied convex membership function types, which are the singleton, the triangle, the trapezoidal, the piece-wise linear, the bell-shaped (Gauss), and the LR type.

### 3.1. SINGLETON TYPE MEMBERSHIP FUNCTION

The α-cut computation is the simplest in case of the singleton type fuzzy sets (see figure 1.a), because one needs to know only the value of the parameter *a*. Here the membership function is described by

$$\mu_{Singleton}(x; a) = \begin{cases} 0 & x \neq a \\ 1 & x = a. \end{cases} \tag{5}$$

All cut endpoints are identical with the value *a*.

### 3.2. TRIANGLE SHAPED, TRAPEZOIDAL AND CONVEX PIECE-WISE LINEAR TYPE MEMBERSHIP FUNCTIONS

In case of triangle shaped, trapezoidal and convex piece-wise linear type membership functions the α-cut computations are similar, thus we discuss these cases together. First of all let us give a brief description of the formulas used for their calculation. We can describe the triangle shaped membership function (see Fig. 1.b) by

$$\mu_{triangle}(x;a,b,c)=\max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right), \tag{6}$$

where the parameters $a$, $b$, and $c$ define the break-points of the shape. Similarly the trapezoidal type membership function (see Fig. 1.c) is defined by

$$\mu_{trapezoid}(x;a,b,c,d)=\max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right), \tag{7}$$

where the parameters $a$, $b$, $c$ , and $d$ define the break-points of the shape. The $i^{th}$ line segment of a convex piece-wise linear membership function (Figure 1.d) is given by

$$\mu_{pwl}(x;p_i,p_{i+1})=\mu_{p_i}+\left(x-x_{p_i}\right)\cdot\frac{\mu_{p_{i+1}}-\mu_{p_i}}{x_{p_{i+1}}-x_{p_i}},\ x\in\left[x_{p_i},x_{p_{i+1}}\right]. \tag{8}$$

where $p_i=\{x_i,\mu_i\}$ and $p_{i+1}=\{x_{i+1},\mu_{i+1}\}$ are the bounding points of the line segment.

In case of the membership function types (6)-(8) the computation of the α-cuts is based on similar triangles. A third point is assigned to the two endpoints of the line segment (Fig. 2) in order to form a rectangular triangle. $p_i$ and $p_{i+1}$ are adjacent points where the $x$ and $\mu$ values (co-ordinates) are known. The sides of the triangle can be calculated by their help.

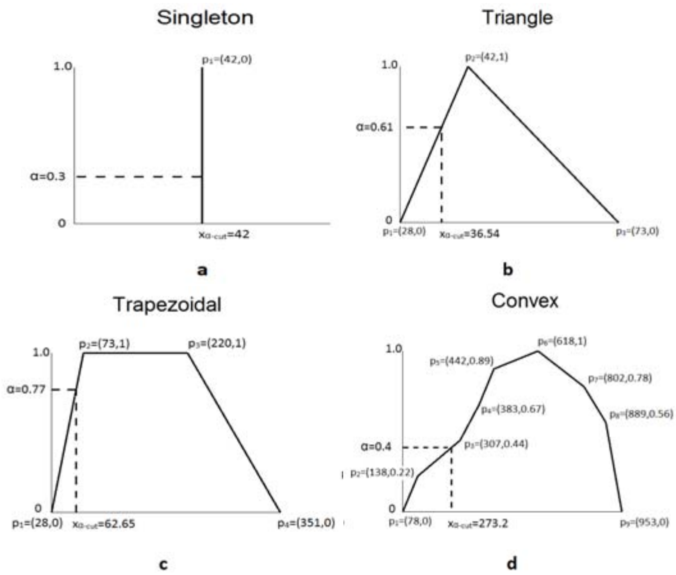


Figure 1 Singleton (a), triangle shaped (b), trapezoidal (c), convex piece-wise linear (d) membership functions and the lower endpoints of their α-cuts

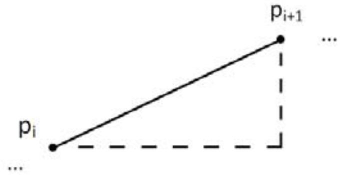


Figure 2. The $i$th linear segment of the set shape

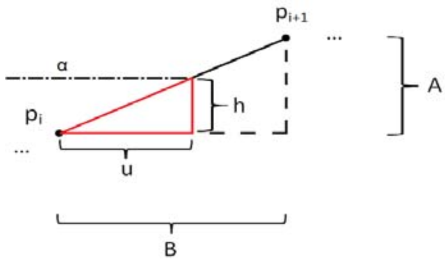


Figure 3. Similar triangles

The determination of a left endpoint of an α-cut is shown in Fig. 3 (the computation of the right endpoint is similar). The figure also shows that the α-cut creates a new triangle. The two existent triangles are similar ones. Their most important feature is that the corresponding sides are in the same ratio. The α-cut computation becomes straightforward owing to this feature.

Fig. 3 shows that the two known sides of the first triangle are $A$ and $B$. The corresponding sides of the triangle created by the α-cut are $h$ and $u$, where the size of $h$ is known. Our task is to determine $u$. In case of the α-cut's left endpoint the $x$ co-ordinate of $p_i$ plus the size of $u$ give the endpoint we are looking for (in case of the α-cut's right endpoint $u$ is subtracted from the abscissa of the $p_{i+1}$ point). For the computation of $u$ the following equations are used.

$$\frac{A}{h}=\frac{B}{u}. \tag{9}$$

$$u=\frac{B\cdot h}{A}. \tag{10}$$

### 3.3. SMOOTH MEMBERSHIP FUNCTIONS

In case of smooth membership functions (e.g. Gaussian, LR, etc) the α-cut computations are similar, thus we discuss these cases also together. An example of a Gaussian type membership function is shown in Fig. 4. It is calculated by the formula:

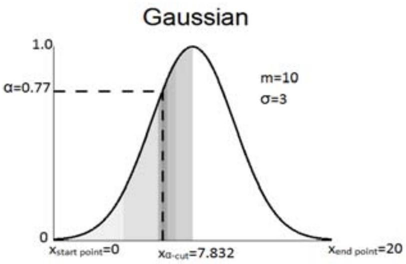


Figure 4 Gauss type membership function and the lower endpoint of its α-cut

$$\mu_{Gauss}(x;\sigma,m)=e^{-\frac{(x-m)^2}{2\sigma^2}},\quad x\in[x_{start},x_{end}] \tag{11}$$

where σ is the variance and $m$ is the expected value, and $x_{start}$, $x_{end}$ are the lower repective upper bounds of the partition.

One calculates the LR function by

$$\mu_{LR}(x;\alpha,\beta,c)=\begin{cases}\sqrt{\max\left(0,1-\left(\frac{c-x}{\alpha}\right)^2\right)} & x<c\\ e^{-\left|\frac{x-c}{\beta}\right|^3} & x\geq c\end{cases},$$

$$x\in[x_{start},x_{end}]. \tag{12}$$

Here we use the bisection method for the calculation of the endpoints of the cuts. After bisecting an interval, one calculates the membership value for the resulted abscissa value ($x$) using the equation (11) or (12). One continues the search in that half interval which contains the demanded α-value. Fig. 4 illustrate the steps of the algorithm. After each bisection one chooses the darker half interval containing the α value. The stopping condition of this method is the execution of 100 iterations, which usually provides a sufficiently good approximation.

## 4. CONCLUSIONS

The presented calculation methods were implemented in C# in form of a dynamic link library (DLL). The lower and upper endpoints of the α-cuts can be calculated by calling the *AlphaCut* method. It takes as parameters the membership function type (we defined an enumeration type for this purpose), the actual parameters of the shape, two array containing the α-levels for which the lower and upper cut-endpoints have to be calculated, as well as two references for the two arrays in which the results will be returned.

We applied the toolbox successfully in course of the development of the software support for a fuzzy arithmetic based student evaluation method (FUSBE) [5], as well as in course of the implementation of an α-cut based fuzzy rule interpolation method called LESFRI [7].

Further research plans include the consideration of other possible quicker algorithms for the calculation of the α-cuts in case of smooth membership function types.

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