SURVEY ON VARIOUS INTERPOLATION BASED FUZZY REASONING METHODS

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Abstract. *Approximate fuzzy reasoning methods serves the task of inference in case of fuzzy systems built on sparse rule bases. This paper is a part of a longer survey that aims to provide a qualitative view through the various ideas and characteristics of interpolation based fuzzy reasoning methods. It also aims to define a general condition set for fuzzy rule interpolation methods brought together from an application-oriented point of view. The methods being presented also can be applied in the first level of systems built on hierarchical fuzzy rule bases.*

Keywords: interpolative fuzzy reasoning, general conditions on rule interpolation methods, sparse fuzzy rule base

1. Introduction

Approximate reasoning methods play an important role in fuzzy logic inference systems. They are required in the case of so-called sparse rule bases. The sparse attribute denotes that the antecedent universes contain at least one partition that according to [13] can be characterized by the formula (1.1).

$$\text{supp}\left(\bigcup_{k=1}^{n} A_{ik}\right) \neq X_i \tag{1.1}$$

where X_i is the ith input universe, A_{ik} is the kth set of the partition of X_i and supp is the support.

With other words in the sparse case the rules do not cover all the input universes whereupon for some observations no rule exists whose premise would overlap the observation at least partially. Essentially a sparse rule-base takes its origin from one of the three reasons specified below:

1. The rules generated from information obtained from experts or from other sources (e.g. neural network-based learning techniques) do not cover all the possible observation values.

2. Gaps between the fuzzy sets can be arisen during the fine-tuning of the system due to the modification of the shape and position of membership functions (Fig. 1.).
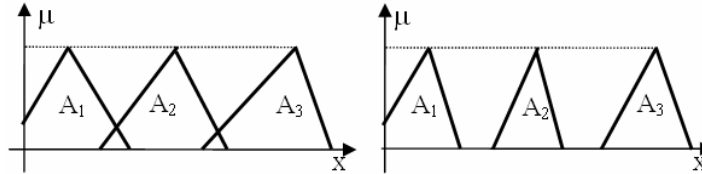

Figure 1. Before and after the fine tuning

3. The number of the state variables is so high that even if all the possible rules can be found out they could not be stored under the given hardware conditions. Taking no notice of the conditions mentioned above the number of the rules grows on. The great number of the rules increases the duration of the inference, too. Thus the performance of the system is decreasing. Making a rule-base sparse artificially [9] or/and transforming it into a hierarchical one (e.g. [26, 27]) could be a possible solution for such cases.

The classical inference methods (e.g. compositional rule of inference) methods are not able to produce an output for the observations covered by none of the rules. That is why the systems based on a sparse rule base should adopt inference techniques, which can perform approximate reasoning taking into the consideration the existing rules. The most applied used methods for this purpose are called interpolative methods.

2. General conditions on rule interpolation methods

A unified condition system related to the interpolative methods would make the evaluation and comparison of the different techniques based on the same fundamentals possible. However, according to the existing literature (e.g. [1, 7, 20, 21, 22]) can be found only partly consistent conditions and condition groups, which are put together taking different points of view into consideration. Therefore, as a step towards the unification, the conditions considered to be the most relevant ones from the application-oriented aspects are going to be reviewed and based on them, some of the well known methods are going to be compared in the followings.

General conditions on rule interpolation methods:

1. *Avoidance of the abnormal conclusion* [1, 7, 20]. The estimated fuzzy set should be a valid one. This condition can be described by the constraints (2.1) and (2.2) according to [20].

$$\inf\left\{B_\alpha^*\right\} \le \sup\left\{B_\alpha^*\right\} \qquad \forall \alpha \in [0,1] \tag{2.1}$$

$$\inf\left\{B_{\alpha 1}^*\right\} \le \inf\left\{B_{\alpha 2}^*\right\} \le \sup\left\{B_{\alpha 2}^*\right\} \le \sup\left\{B_{\alpha 1}^*\right\} \qquad \forall \alpha_1 < \alpha_2 \in [0,1] \tag{2.2}$$

where inf and sup are the lower and upper endpoints of the actual α-cut of the fuzzy set.

2. *The continuity of the mapping between the antecedent and consequent fuzzy sets* [1, 7]. This condition indicates that similar observations should lead to similar results.

3. *Preserving the "in between"* [7]. If the antecedent sets of two neighbouring rules surround an observation, the approximated conclusion should be surrounded by the consequent sets of those rules, too.

4. *Compatibility with the rule base* [1, 7]. This means the condition on the validity of the modus ponens, namely if an observation coincides with the antecedent part of a rule, the conclusion produced by the method should correspond to the consequent part of that rule.

5. *The fuzziness of the approximated result*. There are two opposite approaches in the literature related to this topic [22]. According to the first subcondition (5.a), the less uncertain the observation is the less fuzziness should have the approximated consequent [1, 7]. With other words in case of a crisp observation the method should produce a crisp consequence. The second approach (5.b) originates the fuzziness of the estimated consequent from the nature of the fuzzy rule base [20]. Thus, crisp conclusion can be expected only if all the consequents of the rules taken into consideration during the interpolation are singleton shaped, i.e. the knowledge base produces certain information from fuzzy input data.

6. *Approximation capability* (*stability* [e.g. 21]). The estimated rule should approximate with the possible highest degree the relation between the antecedent and consequent universes. If the number of the measurement (knot) points tends to infinite, the result should converge to the approximated function independently from the position of the knot points.

7. *Conserving the piece-wise linearity* [1]. If the fuzzy sets of the rules taken into consideration are piece-wise linear, the approximated sets should conserve this feature.

8. *Applicability in case of multidimensional antecedent universe*.

9. *Applicability without any constraint regarding to the shape of the fuzzy sets*. This condition can be lightened practically to the case of polygons, since piece-wise linear sets are most frequently encountered in the applications.

3. Surveying some interpolative methods

The techniques being reviewed can be divided into two groups relating to their conception. The members of the first group produce the approximated conclusion from the observation directly. The second group contains methods that reach the target in two steps. In the first step they interpolate a new rule that antecedent part at least overlaps the observation. The estimated conclusion is determined in the second step based on the similarity between the observation and the antecedent part of the new rule.

Further on mostly the case of the one-dimensional antecedent universes are presented for the sake of easy understanding of the key ideas of the methods. As several methods need the existence of two or more rules flanking the observation, therefore it is assumed that they exist and are known. The methods are not based on the same principles, hence sometimes they approach the topic of the rule interpolation from different viewpoints.

3.1. The linear interpolation introduced by Kóczy and Hirota and the derived methods

The first subset of the methods producing the approximated conclusion from the observation directly contains the technique introduced by Kóczy and Hirota and those ones that have been derived from it aiming its extension and improvement. First the most famous member of this group, the KH interpolation is reviewed.

3.1.1. KH interpolation

The key idea of the method developed by Kóczy and Hirota [9] is that the approximated conclusion divides the distance between the consequent sets of the used rules in the same proportion as the observation does the distance between the antecedents of those rules (3.1). This is the fundamental equation of the fuzzy rule interpolation [1] (FEFRI). The proportions are set up separately for the lower and upper distances in the case of each α-cut.

The development of the KH method was made possible by the definition of the fuzzy distance [8] and the fact that fuzzy sets can be decomposed into α-cuts, the calculations can be made by the α-cuts and the conclusion sets can be composed from the resulting α-cuts (resolution and extension principle).

$$d_\alpha^i\left(A_1, A^*\right) : d_\alpha^i\left(A^*, A_2\right) = d_\alpha^i\left(B_1, B^*\right) : d_\alpha^i\left(B^*, B_2\right) \qquad (3.1)$$

where A_1, A_2 are the antecedent sets of the two flanking rules, A* is the observation, B_1, B_2 are the consequent sets of those rules, B* is the approximated conclusion, i can be L or U depending on lower or upper type of the distance. The technique adopted for the determination of the consequent is an extension of the classic Shepard interpolation [16] for case of the fuzzy sets. The method requires the following preconditions to be fulfilled: the sets have to be convex and normal with bounded support, and at least a partial ordering should exist between the elements of the universes of discourses. The latter one is needed for the definition of the fuzzy distance.

The most important advantage of the KH interpolation is its low computational complexity that ensures the fastness required by real time applications. Its detailed analysis e.g. [11, 12, 17] led to the conclusion that the result can not be interpreted always as a fuzzy set, because by some α-cuts of the estimated consequent the lower value can be higher than the upper one (Fig. 2.). The above listed publications defined application conditions that enabled the avoidance of the abnormal conclusion.
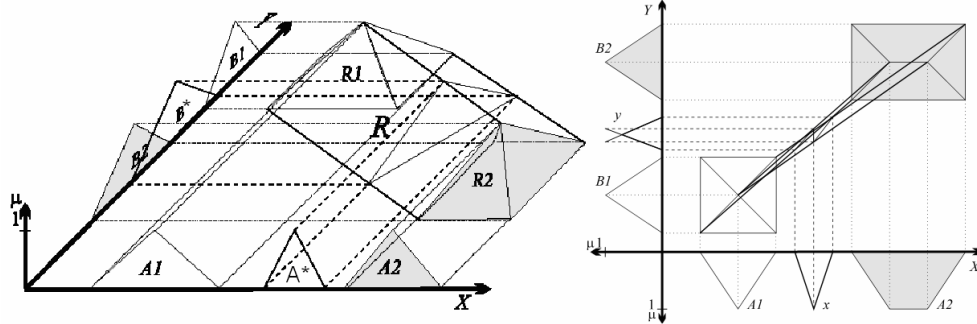

Figure 2. KH interpolation

Theoretically, an infinite number of α-cuts are needed for the exact result if there are no conditions related to the shape of the sets. However, in practice driven by need for efficiency mostly piece-wise linear generally triangle shaped or trapezoidal sets can be found, because these can be easily described by a few characteristic points. Thus supposing the method preserves the linearity completing the calculations for a finite small number of α-cuts could be enough. Although the preceding assumption is not fulfilled, in most of the applications it does not matter because of the negligible amount of the deviation [11, 12, 20].

The KH method was developed for one-dimensional antecedent universes. However, it can be applied in multi-dimensional case using distances calculated in Minkowski sense. It can be simply proven that this technique fulfils *conditions* 3, 4, 5.b and 8. The stabilized (general) KH interpolation [21] also satisfies the condition 6.

The recognition of the shortcomings of the KH interpolation has led to the development of many techniques, which modified or improved the original one or offered a solution for the task of the interpolation using very new approaches. Further on some methods improving the KH technique are reviewed emphasizing those properties which are considered by the authors to be the most important.

3.1.2.   Extended KH interpolation

Several versions of the KH interpolations were developed which allow taking into consideration more than two rules during the determination of the consequence. Their common feature is that the approximation capability of the technique is getting better with the growth of the number of the rules taken into consideration.

In [9] a technique is proposed that takes into consideration the rules weighted with e.g. the reciprocal value of the square of the distance. This approach reflects that the rules situated far away from the observation are not as important as those ones in the neighbourhood of the observation.

The authors of [21] suggest using formulas for the calculation of endpoints of α-cuts of the approximated consequence, which contain the distance on the nth power, where n is number of the antecedent dimensions.

3.1.3.   The VKK method

The method developed by Vass, Kalmár and Kóczy [23] worked out the problem of abnormal conclusion introducing modified distance measures, namely the central distance and width ratio. However, it cannot be applied in case of some crisp sets. Like the KH method it does not conserve the linearity, but the deviance can be proven to be negligible [1].

3.1.4.   Interpolation by the conservation of fuzziness (GK method)

The starting point of the method introduced by Gedeon and Kóczy [3] is that in many applications the supports of the antecedent sets are much more larger than the support of the observation. In such cases the significant feature of the observation is its distance from the nearest flanks of the neighbouring antecedent sets [13].
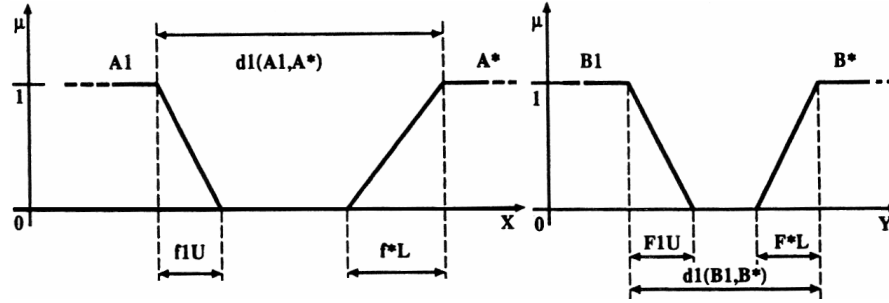
Figure 3. Distance and fuzziness measures [13]

The method was developed for the case of convex normal trapezoidal (incl. triangle shaped and crisp) fuzzy sets. It measures the distance of the sets by the Euclidean distances among the cores (d1(A1,A*) on Fig. 3.). In multidimensional case the Euclidean sum of the distances measured in each dimension is considered. The technique introduces the term of fuzziness of a set (f1U, f*L, F1U and F*L), which is a quantity calculated separately for the left and right flank of the set as the horizontal distance of the respective endpoint of the support and the respective endpoint of the core.

During the interpolation of the conclusion (B*) the flanks are determined by calculating their fuzziness (F*L and F*U). The applied formulas take into consideration the fuzziness of the observation, the distances to the neighbouring antecedent and consequent sets and the neighbouring fuzziness of those sets. The farther sides of the flanking sets are not taken into consideration according to the principle that the interpolated conclusion should be based on "nearby" information [3]. The core points of the approximated conclusion are determined by simple linear interpolation between the nearest core points of the flanking antecedent and consequent sets.

Although the method is not an α-cut based one and has no direct connection with the FEFRI still it is presented in this group of techniques because the way of determining the estimated conclusion is in full accordance with the FEFRI [13]. The GK interpolation is conservative with respect to the degree of local fuzziness in the rule base [3]. On the basis of the literatures [3, 13] it can be stated that the method fulfils the *conditions* 1, 3, 5.a and 8.

3.1.5. Interpolation by the conservation of relative fuzziness (KHG method)

Kóczy, Hirota and Gedeon introduced a refined version of the GK method in [13]. This interpolation technique is in fully accordance with the FEFRI. It is also applicable in case of arbitrary shaped convex and normal fuzzy sets and in such

crisp cases when the use of its ancestor is not possible [13]. It is extended for the multiple dimensional cases, too.
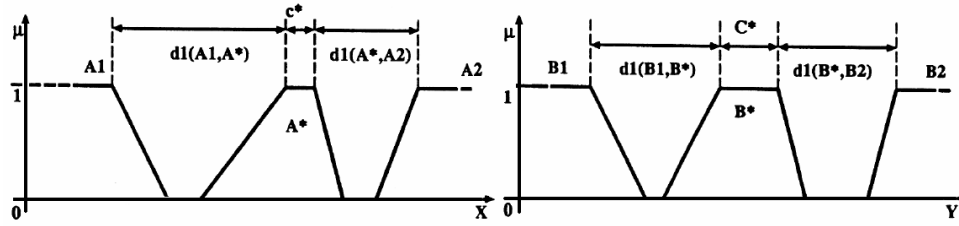

Figure 4. Antecedent and consequent distances and core lengths [13]

The length of the core of the conclusion (C*) (Fig. 4.) is calculated by multiplying the core length of the observation (c*) by the ratio of the distances of the consequent (d1(B1,B2)) and antecedent sets (d1(A1,A2)). The position of the core is determined by the FEFRI introducing a so-called dissimilarity measure. The latter characterises the relation between two lengths, namely a fuzziness value and a distance between two fuzzy sets. The conservation of the relative fuzziness means that the left (right) fuzziness of the approximated conclusion in proportion to the flanking fuzziness of the neighbouring consequent should be the same as the (left) right fuzziness of the observation in proportion to the flanking fuzziness of the neighbouring antecedent. On the basis of the literatures [3, 13] it can be stated that the method fulfils the *conditions* 1, 3, 5.a and 8.

3.1.6.   *Modified α-cut based interpolation*

The modified α-cut based interpolation (MACI) [20] represents each fuzzy set by two vectors describing the left (lower) and right (upper) flank using the technique published by Yam [25]. The vectors contain the break points in case of piece-wise linear membership functions or endpoints of predefined (usually uniform distributed) α-cuts in case of smooth membership functions. The graphical representation of the vectors describing the right flanks of the sets can be seen on the figure 5. The antecedent and consequent sets are represented separately. The result will fulfil the *condition* 1 if B* is situated inside of the rectangle and above of the line l. This goal is reached through a coordinate transformation where Z_0 is substituted by the line *l*. The approximated conclusion will be crisp only if the consequent sets of the rules taken into consideration are singletons, as well.
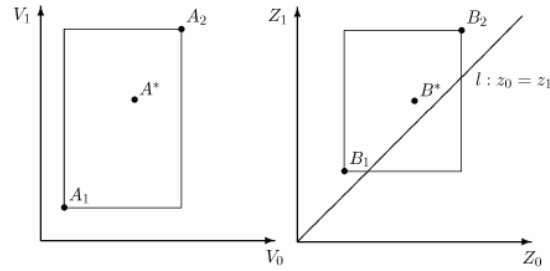
Figure 5. Graphical representation of the vectors [22]

Although this method is not conserving the linearity, the deviance is smaller than in the case of the KH interpolation [20] and the stability experienced at the KH method [19] remains. The estimated conclusion always yields fuzziness if the consequent sets of the rules taken into consideration have fuzziness [15]. The method can be used in multi-dimensional case, too [20]. It can be proven that the technique fulfils the *conditions* 1-4, 5.b, 6, 8 and 9 with the constraint that the sets should be convex and normal. Its generalized version [18] can be used in case of non-convex fuzzy sets, too.

3.1.7.   *The improved multidimensional modified α-cut based interpolation*

The improved multidimensional modified α-cut based interpolation (IMUL) introduced by Wong, Gedeon and Tikk [24] combines the advantages of the MACI and the fuzziness conservation technique proposed by Kóczy and Gedeon in [3]. This method was developed for the case of multidimensional antecedent universe. The fuzzy sets are described by vectors containing the characteristic points, and the coordinate transformation introduced by MACI is used during the determination of the core of the approximated consequent.

The fuzziness of the observation plays a decisive role in the calculation of the flanking edges and beside this the relative fuzziness of the sets adjacent to the observation and adjacent to the approximated consequent are taken into consideration, as well. It can be proven that the technique fulfils the *conditions* 1-4, 5.a, 6, 8 and 9.

3.1.8.   *The HCL interpolation*

The interpolation developed by Hsiao, Chan and Lee (HCL) [4] for the case of triangle shaped convex and normal fuzzy sets combines the KH method with the interpolation of the slopes of the flanking edges.

The basic idea is that the slopes of the approximated conclusion can be calculated with the same linear combination of the respective (left or right) slopes of the

consequents of the neighbouring rules as the linear combination which describes the relation between the respective flanking edges of the antecedents of the same rules and the flanking edges of the observation.

The method produces the estimated conclusion in three steps. First the two endpoints of the support are determined by means of the KH interpolation. After this the peak point of the triangle is calculated employing the relation between the slopes presented above.

The HCL interpolation cannot be classified clearly as an α-cut based technique because it is not based on the resolution and extension principles. It uses only one (usually α=0) α-cut during the calculations. Its advantage is that it results a valid (interpretable) convex and normal fuzzy set having a little higher computational complexity than the KH method.

As a disadvantage can be mentioned that it is applicable only for the case of triangle shaped convex and normal fuzzy sets, not even crisp sets are allowed. Another drawback is the restriction expressing that the same linear combination have to describe on the left and the right side the relation between slopes of the respective edges of the antecedent sets and the slope of the respective edge of the observation. It can be proven that the method satisfies the *conditions* 1, 3, 4 and 5.b.

3.2. Fuzzy interpolation in the vague environment

The fuzzy interpolation in the vague environment (FIVE) introduced by Kovács and Kóczy [e.g. 14] puts the problem of rule approximation in a virtual space in the so-called vague environment whose conception is based on the similarity (indistinguishability) of the objects. The similarity of two fuzzy sets in the vague environment is defined by their distance weighted with the so-called scaling function, which characterizes the vague environment. The scaling function describes the shapes of all the terms in a fuzzy partition.

The challenge during the employment of this method is to find approximate scaling functions for both the antecedent and the consequent universes, which give good descriptions in case of non-Ruspini partitions, too. Scaling functions for the case of triangle and trapezoid shaped fuzzy sets are given in [14]. In consequence of the creation of the vague environments of the antecedent and consequent universes, the vague environment of the rule base is established, as well. In this environment each rule is represented by a point. If the observation is a crisp set, the conclusion, which will be crisp, can be also determined employing any interpolative or approximate technique.

The possibility of creation of the antecedent and consequent vague environments in advance ensures the fastness and hereby the applicability of the method for real-time tasks. Thus, only the interpolation of the points describing the rule base has to be made during the functioning of the system. In case of fuzzy observations the antecedent environment should be created taking into consideration the shape of the set, which describes the input.
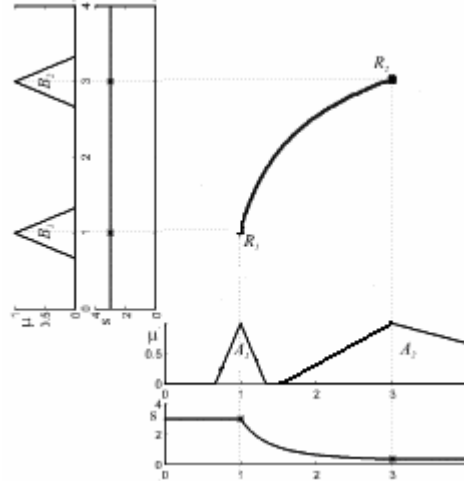

Figure 6. FIVE

Figure 6. presents the partitions, the scaling function and the curve built from the points defined by the existent two rules and the points interpolated for the case of a one dimensional antecedent universe supposing crisp observations. It can be proven that the method satisfies the *conditions* 1-4, 5.a, 6 and 8.

3.3. The generalized methodology

Baranyi, Kóczy and Gedeon proposed in [1] a generalized methodology for the task of the fuzzy rule interpolation. In the centre of the methodology stands the interpolation of the fuzzy relation. A reference point, which can be identical with e.g. the centre point of the core, is used for the characterization of the position of fuzzy sets. The distance of fuzzy sets is expressed by the distance of their reference points. The interpolation consists of two steps.

In the first step an interpolated rule is produced, whose antecedent part has at least a partial overlapping with the observation and whose reference point has the same abscissa as the reference point of the observation. This task is divided into three stages. First with the help of a set interpolation technique the antecedent of the new rule is produced. Next the reference point of the conclusion is interpolated going

out from the position of the reference points of the observation and the reference points of the sets involved in the rules taken into consideration. The applied technique can be a non-linear one, too. Hereupon the consequent set is determined similarly to the antecedent one. Several techniques are suggested in [1] for the task of set interpolation (e.g. SCM, FPL, FVL, IS-I, IS-II). In this paper the solid cutting method is presented in section 3.3.1. If λ_a denotes the ratio, in which the reference point of the observation divides the distance between the reference points of the neighbouring sets into two parts and λ_c denotes the similar ratio on the consequent side, the function $\lambda_c=f(\lambda_a)$ defines the position of the reference point of the consequent set. Through the selection of the function f() a whole family of linear ($\lambda_c=\lambda_a$) and non-linear interpolation techniques can be derived. This is also a possibility for parameterisation (tuning) of the methodology, which ensures the adaptation to the nature of the modelled system.

The approximated rule is considered as part of the rule base in the second step. The conclusion corresponding to the observation is produced by the help of this rule. As the antecedent part of the estimated rule generally does not fit perfectly to the observation, some kind of special single rule reasoning is needed. Several techniques are suggested in [1] for this task (e.g. FPL, SRM-I, SRM-II). As a precondition for all of these methods, it should be mentioned that the support of the antecedent set has to coincide with the support of the observation. Generally this is not fulfilled. In such cases the fuzzy relation (rule) obtained in the previous step is transformed first, in order to meet this condition. For this task, in section 3.3.2, a solution is presented, which was originally suggested in [1].

Owing to the modular structure of the methodology in both of the steps one can choose from many potential methods if some conventional elements (e.g. distance measure) are used consequently. Based on the analysis in [1] and [15] the methodology can be characterized as follows. *Conditions* 1-4, 5.a and 8 are satisfied applying any of the suggested methods. In case of triangle shaped fuzzy sets the *condition* 7 is also fulfilled by those techniques. *Condition* 9 is also satisfied if SCM or FPL is used in the first step and FPL is used in the second step.

3.3.1.   *The solid cutting method*

The key idea of the solid cutting method (SCM) [2] developed by Baranyi et al. is to define vertical axes at the reference points of the two antecedent sets (A_1 and A_2) that flank the observation (A^*) and after that to rotate these sets by 90º around the vertical axes. The virtual space created in such a manner is determined by the orthogonal coordinate axes S, X and μ. The rotated sets will be situated in parallel plane to the plane μxS (Fig. 7.).
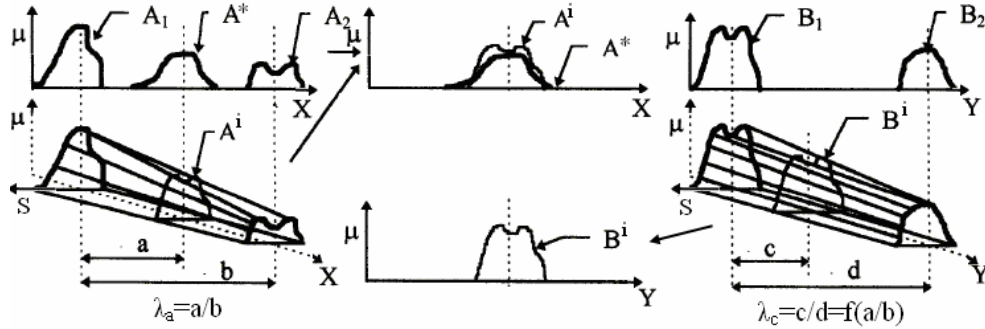
Figure 7. SCM [2]

In the next step a solid is generated fitting a surface on the contour and support of the sets. After this the solid is cut by the reference point of the observation with a plane parallel with $\mu \times S$. Turning back the cross section by 90º one will obtain the antecedent set (A^i) of the estimated rule. The consequence (B^i) of the new rule is determined similarly by knowing the two consequent sets and the reference point.

3.3.2. Single rule reasoning based on Transformation of the Fuzzy Relation and the Fixed Point Law

As mentioned in section 3.3, the support of the antecedent set (A^i) of the interpolated rule does not overlap generally with the observation (A^*). Therefore, the second step of the generalized methodology breaks down into two stages usually. First e.g. the technique "Transformation of the Fuzzy Relation" (TFR) transforms (stretches or shrinks) the interrelation area [1] of the new rule proportionally in order to ensure the needed coincidence of the supports. Secondly the transformed rule is fired applying e.g. the "Fixed Point Law" (FPL).

The TFR transforms the antecedent (A^i) and consequent (B^i) sets separately, but in similar way. Further on only the transformation of the set A^i is presented. First an interrelation function [1] is generated between the observation and the antecedent set in such manner that the endpoints of the support of A^* are mapped to the endpoints of the support of A^i and the reference point of A^i is mapped to the reference point of A^* (Fig. 8.). The rectangle defined by the endpoints of the supports of the sets is called the interrelation area.
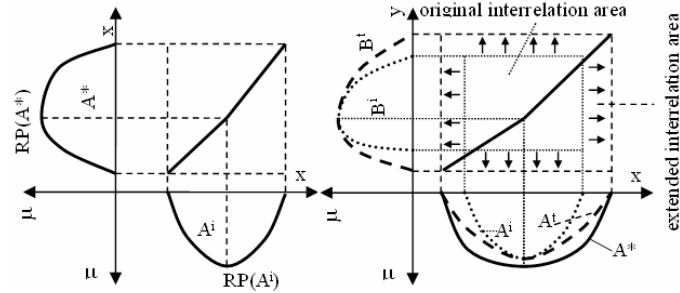
Figure 8. The two interrelation functions

The interrelation function is considered piece-wise linear containing two lines that connect the three characteristic points defined above. Next an interrelation function is generated, which describes the mapping between the points of the two sets (A^i and B^i) participating in the interpolated rule. The aim of the first stage is to modify proportionally the interrelation area of this mapping in such manner to reach the coincidence between the support of the observation and the horizontal side of the rectangle. So the support of the transformed set (A^t) is going to be the same as the support of A^*. The membership value of each point in A^t is equal to the membership value of its interrelated point in A^i.

In the second stage an interrelation function is generated between A^* and A^t similar to the interrelation function defined in the first stage. Next, following the ideas of FPL the difference between the membership values of each interrelated point pair is calculated. This deviation is used by the determination of the approximated conclusion from the transformed consequent B^t taking into consideration the interrelation between A^t and B^t.

3.4. Interpolation with generalized representative values

The method IGRV proposed by Huang and Shen [6] follows an approach similar to the generalized methodology. In the first phase, a representative value (RV) is determined for each used set. Its task is the same as the function of the reference point in the generalized methodology. It can be calculated by different formulas depending on the demands of the application. The centre of gravity played this role in the first variant of the method [5], which was developed for triangle shaped fuzzy sets. In case of an arbitrary polygonal fuzzy set the weighted average of the x coordinates of the node (break) points is suggested as RV. The definition mode of the RV influences only the position of the estimated rule, but not the shape of the sets involved in the rule. Further on the Euclidean distance of the RVs of the sets are considered as the distance of the sets.

The antecedent of the approximated rule is determined by its α-cuts in such a manner that two conditions have to be satisfied. First its representative value has to coincide with the RV of the observation. Secondly the endpoints of the α-cuts of the observation have to divide the distance of the respective (left or right) endpoints of the α-cuts of the neighbouring sets in such proportion as the representative value of the observation divides the distance of the RVs of these sets. Following the same proportionality principle the RV and the shape of the consequent of the approximated rule are determined.

In the second phase, the similarity of the observation and the antecedent part of the new rule is characterized by the scale and move transformations needed to transform the antecedent set into the observation (Fig. 9.). The method was developed primordially for the case of polygonal shaped fuzzy sets. It is applicable in the case of multidimensional antecedent universes, too. In terms of classification, it can be considered as an α-cut based technique, because the scale and move transformation ratios are calculated for each level corresponding to node (break) points of the shape of sets.
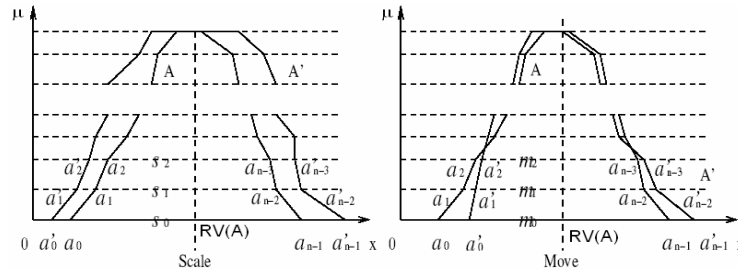

Figure 9. Scale and move transformations [6]

The method is well applicable in case of polygonal shaped sets, but the checking and constraint applications done at each α-level for the sake of the conservation of convexity increase the computational complexity of the technique.

The method can be tuned at two points. First one can choose the formula for the representative value. Secondly, the method for the calculation of the resulting transformation ratios in the case of multidimensional antecedent universes can be chosen. On the grounds of the analysis in [6] it can be stated that the method satisfies the *conditions* 1, 2, 3, 4, 5.a, 8, and 9.

Conclusions

Inference systems based on conventional (compositional) fuzzy inference methods in case of a sparse rule base cannot produce a result for all the possible

observations. In such cases, where the fuzzy rule base could turn to be sparse, the system should adopt an approximate reasoning technique for the estimation of the conclusion. The surveyed fuzzy interpolation methods can be classified into two fundamental groups depending on whether they are producing the result in one or two steps. The first part of this paper gives a brief application oriented survey related to the condition structures can be expected to be fulfilled by the various fuzzy interpolation methods. The second part of the paper enumerates some of the main fuzzy interpolation methods emphasizing their basic ideas, significant characteristics and the conditions they are fulfilling from the above condition structure.

Table 1. Summary of the comparison

Method	1	2	3	4	5.a	5.b	6	7	8	9
KH			X	X		X			X	
Stabilized KH			X	X		X	X		X	
GK	X		X		X				X	
KHG	X		X		X				X	
MACI	X	X	X	X		X	X		X	X^3
Generalized MACI	X	X	X	X		X	X		X	X
IMUL	X	X	X	X	X		X		X	X^3
HCL	X		X	X		X				
FIVE	X	X	X	X	X		X		X	
GM with any techniques	X	X	X	X	X			X^1	X	
GM with SCM/FPL and FPL^2	X	X	X	X	X			X^1	X	X
IGRV	X	X	X	X	X				X	X^3

^{1} only for triangle shaped fuzzy sets

^{2} SCM or FPL in the first step and FPL in the second step

^{3} the sets should be convex and normal

Table 1. contains the brief summary of the conditions that can be considered in accordance with the literature as fulfilled by the studied methods, where the columns represent the conditions, the rows indicate the methods and the cells containing an "X" denote the fulfilled conditions.

This paper has not aimed the presentation of the methods developed especially for hierarchical fuzzy rule bases although they could be very important in case of several practical application types. This topic will be covered by the next part of the survey.

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